Hyperelliptic curves for multi-channel quantum wires and the multi-channel Kondo problem

P. Fendley[1] and H. Saleur[2]
[1] *Physics Department, University of Virginia, Charlottesville VA 22901*
[2] *Department of Physics, University of Southern California, Los Angeles CA 90089-0484*

We study the current in a multi-channel quantum wire and the magnetization in the multi-channel Kondo problem. We show that at zero temperature they can be written simply in terms of contour integrals over a (two-dimensional) hyperelliptic curve. This allows one to easily demonstrate the existence of weak-coupling to strong-coupling dualities. In the Kondo problem, the curve is the same for under- and over-screened cases; the only change is in the contour.

I. INTRODUCTION

In view of the importance of understanding non-fermi-liquid behavior, a wide variety of different one-dimensional models have been studied. Two of the most interesting are the Luttinger liquid and the multi-channel Kondo problem, where a variety of non-perturbative properties have been observed, both theoretically and experimentally.

The methods to study such systems are few. Conformal invariance can be used to understand fixed points and their neighborhoods. In the integrable cases (that turn out to be far more common than could have been hoped for), the Bethe ansatz in principle gives access to all quantities of interest, including the cross-over behavior of Green functions. In practice, many computations are remarkably or impossibly difficult to carry out (see [1,2]; [3,4] for more recent progress), and more direct methods are definitely desired.

Duality could be such a method. In the different context of supersymmetric gauge theories, the last couple of years have witnessed astonishing progress on non-perturbative questions, following the seminal work of [5] where a certain form of duality (combined with analyticity) was first exploited. It seems reasonable to hope that some of these ideas could be used in the context of condensed matter and statistical physics, maybe providing a new, more elegant way of using the Bethe ansatz, and maybe allowing one to solve, at least partially, more general classes of problems.

Some progress in this direction has been accomplished in [6–8], where it was shown that various properties of the Kondo problem at arbitrary spin and the Luttinger tunneling problem did exhibit remarkable representations in terms of hyperelliptic curves. These representations give rise to various forms of duality, and to direct reformulations of the Bethe ansatz in terms of monodromy and differential equations.

Our goal in this note is to extend some of these results to the multichannel case. We will show in particular that an exact duality relation for the current holds in the multichannel quantum wire case, generalizing results of [3], and discuss how Fermi and non-Fermi liquid Kondo fixed points have a unifying representation in terms of contour integrals over hyperelliptic curves.

II. THE MULTI-CHANNEL QUANTUM WIRE

We first consider a problem with k species of electrons (flavors) in one dimension, with a charge interaction and a single impurity. In general, we refer to this problem as a "multichannel quantum wire", though the model might need to be refined to describe experimental situations for general values of k. The case k=1 corresponds to edge states in the fractional quantum Hall effect [9], the case k=2 to quantum wires (with spin isotropy) [10], the case k=4 presumably to armchair nanotubes [11].

Without impurity, the action is made up of a free-fermion part and a charge interaction

$$H = \pi \int dx \left[J_L^2 + J_R^2 + g_{Lutt} J_L J_R\right] \tag{1}$$

where the charge-density current for the left movers is $J_L = \sum_{i=1}^k \psi_{iL}^\dagger \psi_{iL}$, and likewise for right movers. Coupling an impurity to the electrons adds a scattering term

$$\delta H = \int dx\ V(x) \sum_{i=1}^{k} \psi_i^\dagger \psi_i (x=0), \tag{2}$$

where the potential V takes negligible values away from the origin. Like the cases $k=1$ or $k=2$ treated in detail elsewhere [3,12], we can bosonize and perform the usual decomposition into odd and even fields. This yields a theory defined on the full line with a purely chiral interaction at the origin. Calling Θ_i the boson associated with the original fermion ψ_i, we introduce new fields

$$\Phi = \frac{i}{k} \sum_{i=1}^{k} \Theta_i$$

$$\Phi_j = \Theta_j - \frac{1}{k} \sum_{i=1}^{k} \Theta_i. \tag{3}$$

In the bosonic formulation, the fermionic-interaction parameter g_{Lutt} in (1) is replaced by the usual Luttinger parameter g, defined here so that $g = k$ corresponds to the non-interacting case $g_{Lutt} = 0$. The complete action now reads

$$\begin{aligned} S = &\quad \frac{1}{16\pi}\int dx \sum_{j=1}^{k}\left[(\partial_x\Phi_j)^2 + (\Pi_j)^2\right] \\ &+ \frac{k^2}{16\pi g}\int dx\left[(\partial_x\Phi)^2 + (\Pi)^2\right] \\ &+ \lambda\left(e^{i\phi(0)}\sum_{j=1}^{k} e^{i\phi_j(0)} + cc\right) \end{aligned} \tag{4}$$

where ϕ denotes the right-moving component of Φ, and $\sum_{j=1}^{k}\phi_j = 0$. The chiral propagators are

$$\begin{aligned} \langle\phi(z)\phi(w)\rangle &= -\frac{2g}{k^2}\ln(z-w) \\ \langle\phi_i(z)\phi_i(w)\rangle &= -\frac{k-1}{k}\ln(z-w) \\ \langle\phi_i(z)\phi_j(w)\rangle &= \frac{1}{k}\ \ln(z-w), i\neq j \\ \langle\phi_i(z)\phi(w)\rangle &= 0 \end{aligned}$$

In the last sum in (4), we recognize the well-known bosonic expression of the fundamental parafermions χ_1 in the Z_k theory [13,14], so the interaction term can be written equivalently as

$$\lambda\left(e^{i\phi(0)}\chi_1(0) + e^{-i\phi(0)}\chi_1^\dagger(0)\right). \tag{5}$$

One can fold back this theory to obtain what one might call the "level k" generalization of the boundary sine-Gordon model – that is, the boundary version of the generalized supersymmetric, or level k sine-Gordon model well studied in the literature in the bulk case [15]. The case $k = 2$ is the ordinary N=1 supersymmetric sine-Gordon model, and additional details of these various manipulations can be found in [12]. The dimension of the perturbing operator is $d = 1 - \frac{1}{k} + \frac{g}{k^2}$; it is marginal at the non-interacting point $g = k$, and relevant (irrelevant) for $g < k$ ($g > k$).

An interesting way to write the interaction (5) is in the form $J^+ + J^-$, where the J are deformed $SU(2)_k$ currents; this exhibits a relation with the k-channel Kondo model to be discussed in the next section. This is completely analogous to the $k = 1$ case studied in detail in [16,17], where the $J^\pm$ are the usual vertex operators.

It is worthwhile to comment briefly on cocycles (Klein factors) here, which have to be handled with great care in problems with several fermion species. In addition to the usual exponential of a free boson, each fermion $\psi_{i,LR}$ requires a real cocycle $\eta_{i,LR}$ such that $\{\eta_{i,C}, \eta_{j,C'}\} = \delta_{ij}\delta_{CC'}$. In the last term in (4), this means that each exponential $e^{i\phi_j(0)}$ should come up, in fact, with a prefactor $\eta_{jL}\eta_{jR}$. However, because of charge neutrality, and the fact that pairs of fermions commute, these factors disappear from the computation of physical quantities like the free energy or the conductance, and can be safely ignored.

The method of [3] and [12] can be easily generalized to compute the current using Bethe ansatz and massless scattering *. The current at $T = 0$ can be found explicitly by using the Wiener-Hopf method. The formulas of [3] and [12] generalize to the case of k channels, with a k dependent kernel (using the notations of [12])

$$N(\omega) = \sqrt{2\pi(\frac{1}{k} + h')}\frac{\Gamma[i(1+kh')\omega/2h']\Gamma(i\omega/2)}{\Gamma(i\omega/2h')\Gamma(ik\omega/2)\Gamma\left(\frac{1}{2}+\frac{i\omega}{2}\right)}e^{i\omega\Lambda}$$

where to parameterize the interactions we introduce

$$h' = \frac{1}{g} - \frac{1}{k}$$

(h' is denoted $1/\gamma$ in [12]). The parameter g is the conductance without impurity (in units of e^2/h), so $g = k$ for k channels of free electrons.

The interactions are parameterized by $u \propto V\lambda^{1/h}$. where we have defined the coupling h "dual" to h' by

$$h = \frac{g}{k^2} - \frac{1}{k}$$

with $h < 0$. Following [3,12], it is then straightforward to find the weakly-interacting (large u) and strongly-interacting (small u) series expansions of the current for general k. Defining the scaled current $\mathcal{I}_k = I/gV$ as in [7], its UV expansion is

$$\mathcal{I}_k = 1 + \frac{\sqrt{\pi}}{2}\sum_{n=1}^{\infty}(-1)^n\frac{\Gamma(n(1+kh)+1)\Gamma(nh+1)}{n!\Gamma(nkh+1)\Gamma(nh+3/2)}u^{2nh} \tag{6}$$

Similarly, for the IR expansion, one finds

$$\mathcal{I}_k = \frac{\sqrt{\pi}}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\Gamma(n(1+kh')+1)\Gamma(nh'+1)}{n!\Gamma(nkh'+1)\Gamma(nh'+3/2)}u^{2nh'}. \tag{7}$$

One can define a crossover temperature $T_B \propto \lambda^{-1/h}$ analogous to the Kondo temperature T_K †.

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*There are subtle issues about charging effects or the way the voltage difference would be imposed in a real quantum wire, that we intend to discuss elsewhere. In the present paper, V simply controls the difference of populations of left and right movers in formal analogy with the $k = 1$ Hall case.

†The exact relation between T_B and the bare parameter λ can be found within the dimensional regularization scheme usual in the TBA approach, see eg [12]. One finds $u = \frac{V}{T'_B}$, with $T'_B = \frac{N(-i)}{N(0)}e^{-\Delta}T_B$, $T_B = e^{\theta_B}$, and θ_B the usual rapidity variable in the reflection matrix.

Remarkably, it turns out that this current can be written as the integral

$$\mathcal{I}_k(g,u) = \frac{i}{4u}\int_{\mathcal{C}_0} dx \frac{(1+x^h)^{k-1}}{\sqrt{x(1+x^h)^k - u^2}}, \tag{8}$$

where the curve $\mathcal{C}_0$ starts at the origin, loops around the branch point on the positive real axis, and goes back to the origin. We derive the result (8) in the appendix. The current turns out to obey the self-duality relation

$$\mathcal{I}_k(g,u) = 1 - \mathcal{I}_k\left(\frac{k^2}{g}, u\right). \tag{9}$$

This follows directly from the expansions (6,7); it can also easily be proven from the integral (8) by changing variables $x \to x^{g/k}$ on the right-hand side and integrating by parts. The physical origin of the duality is similar to the $k=1$ case [7,18]. Integrability restricts the irrelevant operators near the IR fixed point to be (within a dimensionally regularized scheme) mutually commuting conserved quantities, either neutral, or of the form

$$e^{i\frac{k}{g}\phi}\chi_1 + e^{-i\frac{k}{g}\phi}\chi_1^\dagger. \tag{10}$$

In particular, no harmonics of (10) appear. It can also be shown that the neutral quantities do not contribute to the DC current, which is, in effect, completely determined by (10), hence giving rise to (9).

The representation (8) lets us easily find all the Lee-Yang singularities of the current. Write $\mathcal{I}_k = \int \frac{dx}{y}$. Two or more roots of y coalesce at values u and x where both $y=0$ and where $dy/dx = 0$. The current will have a singularity if the contour runs in between the coalescing roots. For $k>1$, one such value is $u=0$, where k roots coalesce at $x^h = -1$. Since the contour $\mathcal{C}_0$ for the current is trivial when u=0 (it starts at the origin and loops around the root at x=0), the current is not singular. We will see in the next section that the overscreened Kondo problem however has very interesting behavior as a result of these k roots coalescing. All singularities other than u=0 are at magnitude

$$|u_0|^2 = (-h)^k\,(h+1)^{-k-\frac{1}{h}}\,;$$

(recall that h is negative). For real physical values of u, the contour never is singular. The value of $|u_0|$ does give the radius of convergence of the two perturbation expansions; the large-u series converge for $|u|>|u_0|$ and the small u series for $|u|<|u_0|$.

III. THE MULTI-CHANNEL KONDO PROBLEM

As is well known and discussed in depth in [16], the single-channel Kondo problem and the Luttinger liquid with impurity are deeply related. Not surprisingly, the multichannel problems are as well.

The Kondo model describes three-dimensional non-relativistic electrons coupled to a single impurity spin. Considering the radial modes reduces the problem to gapless electrons on the half-line coupled to a quantum-mechanical spin $\mathbf{S}$ at the boundary. In the multichannel Kondo problem, there are k channels of electrons $\psi_{i\alpha}$ where $i = 1\ldots k$ and $\alpha = 1,2$ is the spin index [19]. One can then form an $SU(2)_k$ "spin" current

$$\mathbf{J}(x) = \sum_{i=1}^{k} \psi^\dagger_{i\alpha}\sigma_{\alpha\beta}\psi_{i\beta}$$

using the Pauli matrices σ. One can similarly form $SU(k)_2$ "flavor" currents and a $U(1)$ "charge" current. In conformal field theory language, there are $2k$ Dirac fermions, which can be bosonized in terms of the current algebras $SU(2)_k \times SU(k)_2 \times U(1)$ [20]. The corresponding WZW theories have central charges $3k/(k+2), 2(k^2-1)/(k+2)$ and 1 respectively, adding up to $2k$ as they should.

Since only the spin current couples to the impurity, it is the only one which we need here. Thus just like the multi-channel wire considered in the previous section, the multi-channel Kondo model is associated with $SU(2)_k$. The impurity is represented by a quantum-mechanical spin $\mathbf{S}$ in the spin-S representation. For an impurity located at $x=0$, the fermions are coupled antiferromagnetically via a term in the Hamiltonian

$$\delta H = \lambda \mathbf{J}(0)\cdot\mathbf{S}$$

for positive λ. The coupling λ is dimensionless since the current is of dimension one, but there is a short-distance divergence in perturbation theory in λ. Thus the interaction term is marginally relevant, and a mass scale is present in the theory. In particle-physics language, the Kondo model is asymptotically free and undergoes dimensional transmutation. This scale generated is usually called the Kondo temperature T_K, and it is completely analogous to Λ_{QCD} in gauge theory. In terms of the original parameter λ [19],

$$T_K \sim \lambda^{k/2} e^{-const/\lambda}. \tag{11}$$

The renormalized theory parameter T_K remains finite while the bare parameter $\lambda \to 0$.

As λ gets large (or more precisely, we study physics at energy scales below T_K), the system crosses over to a strongly-coupled phase. At $T_K \to \infty$, there is another fixed point, where the the electrons try to bind to the spin. Because of Pauli exclusion only a single electron from each channel can bind to the impurity. Thus the problem naturally splits into three cases: overscreened ($k>2S$), exactly screened ($k=2S$) and underscreened ($k<2S$). At this strongly-coupled fixed point, the spin of the impurity is effectively reduced to zero in the first

two cases, while it is reduced to $S - k/2$ in the underscreened case.

It is convenient to consider a more general model, the anisotropic Kondo model, which allows for $SU(2)$-breaking interaction $\lambda_z J_z S_z$. As detailed in [16] for the single-channel case, this remains integrable as long as the impurity spin is the appropriate representation of the quantum group $SL(2,q)$. For $S = 1/2$ this distinction is irrelevant, since the representation is identical (the Pauli matrices). As with the quantum wire, we parameterize the anisotropy by the parameter g with $0 < g \le k$, where $g = k$ corresponds to the $SU(2)$-invariant isotropic point for the k-channel problem. After a few simple transformations to gauge away the $J_z S_z$ term and unfolding, the action for the spin degrees of freedom reads much like in the previous section (4), the only difference being the impurity term, which is now of the form

$$\lambda \left(S^- e^{i\phi(0)} \chi_1(0) + S^+ e^{-i\phi(0)} \chi_1^\dagger(0) \right) \tag{12}$$

Here, ϕ is a spin boson (it was rather a charge boson in (4)), and $S^\pm$ are raising and lowering $SL(2,q)$ operators in the appropriate spin j representation. The deformation parameter $q = e^{i\pi h}$, where $h = \frac{g}{k^2} - \frac{1}{k}$ as with the multichannel wire. The perturbing operators are not marginal as in the isotropic case, but are relevant with dimension $d = 1 + h$. An interesting point is the Toulouse point, where for $k = 1$ and $g = 1/2$, or $k = 2$ and $g = 0$ [21], the problem reduces to free fermions and can be solved without recourse to the Bethe ansatz. For a review of the current status of much of the theory and experiment of the multi-channel Kondo model, see [22].

The free energy in the multi-channel Kondo problem was derived using the Bethe ansatz in [23,24]. It is given in terms of the solution of a set of an infinite number of non-linear integral equations. These equations cannot be solved in closed form at arbitrary temperature, but at zero temperature they reduce to a single linear integral equation, which can be solved by the Wiener-Hopf technique. The physical quantity we will study is the magnetization $M_{k,S}$ of the spin-S impurity as a function of applied magnetic field H (the magnetic field couples to the conserved total charge $J_z + S_z$). At zero temperature, the $M_{k,S}$ is a function only of the dimensionless quantity u, where

$$u = \frac{g\Gamma(1/2h')\Gamma(k/2)}{2\pi k^{k/2}\Gamma(k/2 + 1/2h')} \frac{H}{T_K}.$$

In the no-coupling limit, $M_{k,S}(u \to \infty) = S$, while in the strong-coupling limit, $M_{k,S}(0) = 0$ for the overscreened case or exactly screened cases $k \ge 2S$ and $M_{k,S}(0) = (S - k/2)N/g$ for the underscreened cases $k < 2S$.

The entire magnetization for the isotropic case was derived in [23] using the Bethe ansatz. It is straightforward to generalize this to all g; the result for the overscreened case and exactly screened cases $k \ge 2S$ is

$$M_{k,S}(u) = \frac{i}{4\pi^{3/2}} \int_{-\infty}^{\infty} \frac{d\omega}{\omega - i\epsilon} e^{2i\omega \ln(u)} \frac{\sinh(2S\pi\omega)}{\sinh(k\pi\omega)} \times \frac{\Gamma(i\omega)\Gamma(\frac{1}{2} - i\omega)\Gamma(1 - i\omega/h)}{\Gamma(ik\omega)\Gamma(1 - i\omega g/kh)} \tag{13}$$

where ϵ is positive and tending to zero. While this expression is somewhat unwieldy, it is easy to find complete perturbative expansions from it by completing the contour in the upper half-plane for u large, and in the lower half-plane for u small. The poles in the upper half-plane are at $\omega = -inh$ for $n \ge 0$ an integer (recall that $h < 0$). Thus for u large enough (the precise limit will be given below), the magnetization for spin 1/2 is

$$M_{k,1/2} = \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(\frac{1}{2} - nh)\Gamma(1 - knh)}{\Gamma(1 - nh)\Gamma(1 - nkh - n)} u^{2nh} \tag{14}$$

In the isotropic case $h \to 0$, this expansion breaks down because the exponent goes to zero while $u \to 0$ as h^k for fixed H/T_K. In this case, the appropriate expansion involves $\ln(H/T_K)$, as is clear from (11).

The main result of this section is that the magnetization in the k-channel Kondo problem can be expressed simply in terms of a hyperelliptic curve:

$$M_{k,S}(u) = \frac{iu}{4\pi} \int_{\mathcal{C}_S} \frac{dx}{xy} \tag{15}$$

where

$$y^2 = (-1)^{2S} x(1 - x^h)^k + u^2 \tag{16}$$

and the contour $\mathcal{C}_S$ starts at infinity and goes around the "first" 2S branch points. The hyperelliptic curve (16) differs by that in the previous section only by minus signs. This seemingly innocuous change is responsible for the interesting non-fermi-liquid behavior in the overscreened Kondo problem.

The derivation of this curve for $S = 1/2$ starting from the series expansion (14) is similar to that of the appendix, so we omit it here. The contour $\mathcal{C}_{1/2}$ starts at infinity (as opposed to the origin for $\mathcal{C}_0$), loops around the branch point on the real axis for real positive u, and returns to infinity. The derivation of the contours for higher impurity spins is identical to the derivation for $k = 1$ in [6]. Since $S = 1/2$ is the most interesting physical case, and the contours for higher spin were discussed in detail in [6], we give only a brief explanation here. The higher-spin magnetizations follows from the "fusion" relation valid at large u:

$$M_{k,S}(iu) + M_{k,S}(-iu) = M_{k,S-1/2}(u) + M_{k,S+1/2}(u) \tag{17}$$

where the argument iu is meant as the continuous deformation of $u \to iu$ at fixed large $|u|$. For the overscreened case $k > 2S$, this relation is actually valid at all

u. As one can see from the form of the expansion (14), $M(e^{2\pi i}u) \neq M(u)$. The "first" $2S$ branch points are then those obtained by rotating the original branch point on the real axis by $e^{2\pi i j}$ for $j = -(S-1/2)\dots(S+1/2)$. These points are on different sheets because of the branch cut along the negative x-axis due to the x^h in (16). For example, for $S=1$ and large u, on the original sheet there are branch points just above and below the negative real axis near $x \approx -u^2$. Thus when u is large, the contour $\mathcal{C}_1$ starts at infinity, loops around the upper branch point, returns to infinity and then loops around the other branch point. This can be deformed to a closed contour surrounding the two branch points on the original sheet. By analytic continuation, this contour is valid for any value of u, not just u large where the formula (17) applies.

The behavior near the strong-coupling fixed point is easy to find using the curve. Because of the $(1-x^h)^k$ in (16), for u small there are k branch points near $x=1$. As opposed to the current in the previous section, the contour here involves these points.

For the exactly-screened case $k=2S$, the contour surrounds just these k branch points. Expanding the integrand in (15) in powers of u^2 and using the beta-function identity (A1) yields the expansion

$$M_{k,k/2} = \sum_{n=0}^{\infty} a_n u^{2n+1} \tag{18}$$

where

$$a_n = \frac{1}{h\sqrt{\pi}} \frac{(-1)^n \Gamma(-(n+\frac{1}{2})/h)\Gamma(n+1/2)}{n!\Gamma(1-(n+\frac{1}{2})g/kh)\Gamma(k(n+\frac{1}{2}))}$$

good for small u. This of course agrees with the result obtained from completing the contour in (13) in the lower half-plane. Physically, this expansion means that the irrelevant operators near the IR fixed point are all scalar, of even dimensions: powers of the stress energy tensor and the like, exactly as in the $k=1$ case.

The under-screened case is very similar to the single-channel case discussed in [6]. The irrelevant operators near the IR fixed point then involve not only the foregoing scalar operators, but also, like in the tunneling problem, a single charged operator

$$S^- e^{ik\phi/g}\chi_1 + S^+ e^{-ik\phi/g}\chi_1^\dagger, \tag{19}$$

where S now is a $SL(2,q)$ spin in the $j-\frac{k}{2}$ representation. The dimension of this operator is $d = 1 + h' = \frac{k-1}{k} + \frac{1}{g}$. The form (19) establishes a duality between the strong and weak coupling regimes, similar to what happens for $k=1$ [25,6]. This duality is only "partial", because, in contrast to the current in the previous section, the magnetization, when expanded in the IR, depends not only on the term (19), but also on the scalar irrelevant operators.

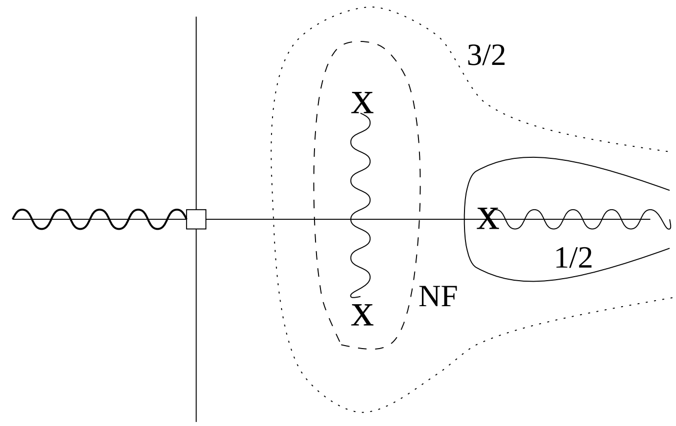

FIG. 1. The contours for spin 1/2 (overscreened) and spin 3/2 (exactly screened) impurities for the three-channel Kondo problem. The three square-root branch points illustrated all approach $x=1$ as $u \to 0$.

The overscreened case is the most interesting because of the non-fermi-liquid behavior even in the isotropic limit $h \to 0$. The curve is singular at $u=0$ because k roots are coalescing at $x=1$. The magnetization is singular as well because the contour runs in between these coalescing roots (for the under-screened and exactly screened cases, the contour surrounds all of these and so is not singular). For the example $k=3$, this is illustrated in figure 1; the roots pictured are those which meet at $x=1$ when $u=0$. This singular behavior when the curve goes in between these roots results in a non-fermi critical exponent. We illustrate this first for odd k. The contour for $S=1/2$ can be written as the sum of two contours, the exactly-screened contour $\mathcal{C}_{k/2}$ plus a contour $\mathcal{C}_{\rm NF}$ surrounding the other $k-1$ branch points near $x=1$. The exactly-screened contour of course yields the expansion (18) with its Fermi-liquid exponent. To find the appropriate expansion for the contour $\mathcal{C}_{\rm NF}$, we change variables in (15) by $r=(1-x^h)u^{-2/k}$, so for odd k

$$M_{k,1/2} = (-1)^{(k-1)/2} M_{k,k/2} + \frac{i}{4\pi h}\int_{\mathcal{C}_{\rm NF}} \frac{dr}{1-u^{2/k}r} \frac{u^{2/k}}{\sqrt{1-r^k(1-u^{2/k}r)^{1/h}}}.$$

This can be expanded in powers of $u^{2/k}$ when u is small, giving the appropriate non-fermi critical exponent [23,24]. To find the coefficients of this expansion, one divides the contour $\mathcal{C}_{\rm NF}$ into $k-1$ contours each starting and ending at the origin $r=0$, and again utilizes a computation similar to the appendix. The result is

$$M_{k,1/2} = (-1)^{(k-1)/2} M_{k,k/2} + \frac{1}{2\pi^{1/2}} \sum_{n=1}^{\infty} \frac{\Gamma(\frac{1}{2}-n/k)\Gamma(1-n/kh)}{n!\Gamma(1-n/k)\Gamma(1-n/kh-n)} u^{2n/k}$$

of course in agreement with the residue expansion of (13). This expansion is still valid in the isotropic limit $h=0$.

These results are interpreted physically as follows. In addition to the scalar quantities, there is another irrelevant operator controlling the approach to the IR fixed point in the overscreened case, and replacing (19), of the form

$$\epsilon_1 \partial\phi \tag{20}$$

where ϵ_1 is the energy field of the Z_k parafermionic theory, of dimension $d = \frac{2}{k+2}$ [13]. There is a slight subtlety concerning the computation of the magnetization near the IR fixed point in the overscreened case, since the impurity spin has disappeared right at the fixed point [20]. At zero temperature, if we call λ_d the coupling of (20), it turns out that the magnetization goes as $(H^d \lambda_d)^{1/(1-d)}$ [20], so indeed the magnetization goes as $u^{2/k}$.

For k even, although the curve is still simple, the small-coupling expansion is cumbersome because there are an even number of roots coalescing. For example, for $k = 2$ and u small, the leading term is

$$M_{2,1/2} \approx \frac{iu}{4\pi h} \int_{1+u}^{const} \frac{dx}{x} \frac{1}{\sqrt{u^2 - (x-1)^2}} \propto u \ln u. \tag{21}$$

The fusion relation (17) lets us see right away that the log terms must be related to the expansion (18) for the exactly screened problem. In general, for k even, the expansion is of the form

$$M_{k,1/2} = (-1)^{(k-1)/2} \sum_{n=1}^{\infty} (-1)^n \frac{a_n}{\pi} u^{2n+1} \ln u + b_n u^{2n/k} \tag{22}$$

where the a_n are given above, and the b_n are quite complicated (involving a sum of digamma functions). The easiest way to find the b_n is in fact to go back to the original Bethe ansatz expression (13).

We finally discuss the isotropic limit $g = k$ ($h \to 0$), which is particularly intriguing. In this limit $u \to 0$ as $h^{k/2}$ for fixed H/T_K. Denoting $\mathcal{M}_{k,S}(H/T_K) = \lim_{g \to 1} M_{k,S}(u)$, its integral form is

$$\mathcal{M}_{k,S} = \frac{i}{4\pi} \int_{\mathcal{C}_S} \frac{dx}{x} \frac{H/T_K}{\sqrt{(-1)^{2S} 2\pi x (\ln x)^k + (H/T_K)^2}}.$$

Thus it is obvious why the weak-coupling perturbation expansion around $T_K = 0$ involves logarithmic terms. To formulate a large H/T_K perturbation theory, we define the parameter

$$\ln(H/T_K) = \frac{1}{z} - \frac{k}{2} \ln(z/4\pi). \tag{23}$$

By a change of variables, the magnetization can be written for $g \to 1$ as

$$M_{k,S} = \frac{i}{4\pi} \int_{\mathcal{C}_S} \frac{dx}{x} \frac{1}{\left[(-1)^{2S} x((z/2)\ln x - 1)^k + 1\right]^{1/2}} \tag{24}$$

As with the single-channel case, this can be expanded in powers of z:

$$\mathcal{M}_{k,S}(z) = \sum_{n=0}^{\infty} \mathcal{A}_n z^n \tag{25}$$

For spin $S = 1/2$ this expansion is asymptotic. It has zero radius of convergence because as $x \to \infty$ the $x(z \ln x)^k$ term will eventually dominate the integral no matter how small z is. In contrast, observe that the multi-channel quantum wire discussed in the previous section is very simple at $h \to 0$; the integral for the current can easily be done explicitly. The perturbing operator (2) for the wire is exactly marginal at this value of h, so the model remains conformally invariant even with the perturbation. Therefore no scale is introduced, and the problem can be solved using the techniques of boundary conformal field theory.

This research was supported by NSF grant DMR-9802813 (P.F.), the NYI program (NSF-PHY-9357207) and DOE grant DE-FG03-84ER40168 (H.S.)

APPENDIX A: THE DERIVATION OF THE CURVE

We consider the expression

$$f(\lambda) = \frac{i}{2} \int_{\mathcal{C}_0} dx \; \frac{\left(1 + \lambda x^h\right)^{k-1}}{\left[x\left(1 + \lambda x^h\right)^k - 1\right]^{1/2}},$$

where the x integral is along the usual contour starting at the origin, looping around the branch point on the real axis when λ is real, and ending at the origin. We represent $f(\lambda)$ as a double integral

$$\frac{i}{2} \int\int dxdz \; \frac{z^{k-1}}{\left(xz^k - 1\right)^{1/2}} \; \delta\left[z - \left(1 + \lambda x^h\right)\right].$$

We can now expand the square root by setting $xz^k - 1 = x - 1 - x(1 - z^k)$, leading to

$$\int\int dxdz \; z^{k-1} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)} \frac{x^n \left(1 - z^k\right)^n}{(1-x)^{n+1/2}} \times \delta\left[z - \left(1 + \lambda x^h\right)\right].$$

We now represent the delta function as a third integral

$$\delta\left[z - \left(1 + \lambda x^h\right)\right] = \int dt \; e^{2i\pi t\left(z - 1 - \lambda x^h\right)},$$

and expand the term $e^{-2i\pi t \lambda x^h}$ in power series. Each x integral is done by using the contour-integral representation of the beta function

$$\frac{\Gamma(a)}{\Gamma(a+b)\Gamma(1-b)} = \frac{i}{2\pi}\int_{\mathcal{C}_0} dx\ x^{a-1}(x-1)^{b-1}, \qquad \text{(A1)}$$

yielding

$$\int dzdt\ e^{2i\pi t(z-1)} z^{k-1} \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \left(1-z^k\right)^n \times \frac{\Gamma(n+ph+1)}{\Gamma(1/2)\Gamma(n+1)\Gamma(ph+3/2)} \frac{(-2i\pi t\lambda)^p}{p!}.$$

To remove the t^p piece, we integrate by parts p times in z. The t integral then yields $\delta(z-1)$, which then lets us do the z integral, yielding

$$\frac{1}{\sqrt{\pi}}\sum_{p=0}^{\infty} \lambda^p\ \frac{\Gamma(ph+1)}{\Gamma(ph+3/2)p!}\ C_p,$$

where

$$C_p = \sum_{n=0}^{\infty} \frac{\Gamma(n+ph+1)}{\Gamma(ph+1)n!}\ \frac{d^p}{dz^p}\ z^{k-1}\left(1-z^k\right)^n\Big|_{z=1}.$$

The sum over n actually truncates above at $n=p$ because of the z=1 limit, and below at $n=(p-k+1)/k$ because of the derivatives. However, it is more convenient to leave the bounds 0 and ∞, because the sum in C_p can then be done:

$$\begin{aligned} C_p &= \frac{d^p}{dz^p}\left(z^{k-1}\sum_{n=0}^{\infty}\frac{\Gamma(n+ph+1)}{\Gamma(ph+1)n!}(1-z^k)^n\right)\Bigg|_{z=1} \\ &= \frac{d^p}{dz^p}\left(z^{k-1}z^{-k(ph+1)}\right)\Big|_{z=1} \\ &= (-1)^p\frac{\Gamma(kph+p+1)}{\Gamma(kph+1)}. \end{aligned}$$

It follows that

$$f(\lambda) = \frac{1}{\sqrt{\pi}}\sum_{p=0}^{\infty}(-1)^p\ \lambda^{2p}\ \frac{\Gamma(ph+1)\Gamma(kph+p+1)}{p!\Gamma(ph+3/2)\Gamma(kph+1)}. \qquad \text{(A2)}$$

Of course, the above manipulations are true only for values of λ where the series converges. Using a straightforward change of variables one finds the final form

$$\mathcal{I} = \frac{i}{4u}\int_{C_0} dx\ \frac{\left(1+x^h\right)^{k-1}}{\sqrt{x\left(1+x^h\right)^k - u^2}}. \qquad \text{(A3)}$$

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